This derivation depends on the following formula given by Jacobi, (Crelle, Vol. 15):

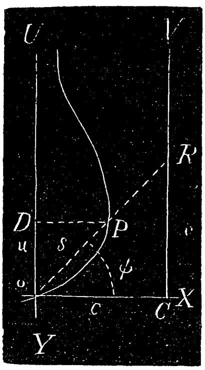
$$\int_o^\pi \phi(\cos t) \cos n\, t \,.\, d\, t = \frac{1}{1.3.5\ldots(2\,n-1)} \cdot \int_o^\pi \phi^{(n)} (\cos t) \sin^{2n} t \,.\, d\, t,$$

where (n) denotes repeated differentiations.

DISCUSSION OF AN EXPONENTIAL CURVE.

BY IRVING P. CHURCH, B. C. E., NEWBURGH, N. Y.

Construction by Points.—Let $O\ C = c$ be the base of any logarithmic system. On $O\ U$ and $C\ V$, perpendicular to $O\ C$, lay off $O\ D = u$, and $C\ R = v$, such that $u = \log v$ in the system whose base is c. The intersection of $O\ R$ and $D\ P$, (parallel to $O\ C$) fixes P, a point of the curve.



From the construction we can immediately derive the polar equation of the curve:

$$O\ P = s = \frac{u}{\sin\phi} = \frac{\log v}{\sin\phi} = \frac{\log.[c\tan.\phi]}{\sin\phi}$$

$$= \frac{\log.c + \log\tan\phi}{\sin\phi} \ \ldots\ldots\ldots\ldots (1)$$

Substituting from $y = s \sin\phi$, and $\tan\phi = \frac{y}{x}$, we have for rectangular co-ordinates,

$$y = \log.c + \log\left(\frac{y}{x}\right) = \log.\left\{\frac{c\,y}{x}\right\} \text{ or } c^y = \frac{c\,y}{x},$$

whence we have the final equation of the curve,

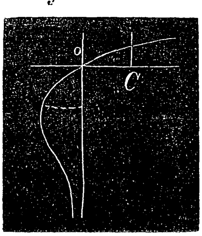
$$x = \frac{y}{c^{y-1}}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and also successively,

$$\frac{dx}{dy} = c^{1-y}(1 - y\ l\ c),$$

$$\text{and } \frac{d^2x}{d\,y^2} = c^{1-y}\ l\ c\ (y\ l\ c - 2);\ \text{here } l\ c = \text{Nap. log}\,.\,c.$$

By reference to the equation (2) we see that if $c >$ unity the curve has the form indicated by our first diagram, which will give a maximum for x; also if $c =$ unity, (2) becomes $x = y$, the equation of a straight line bisecting the first angle, while if $c <$ unity, the curve passes to the other side of the line just mentioned, and is tangent to Y at $y = -\infty$, thus, and gives a minimum for x.



It is now our object to find, when $c > 1$, its particular value, such that $c = x$ max. From

$$\frac{dx}{dy} = 0 \text{ we have } y = \frac{1}{l\,c}, \text{ whence } x \text{ max.} = \frac{\frac{1}{l\,c}}{c^{\left(\frac{1}{lc} - 1\right)}},$$

and putting this equal to c, we have

$$\frac{1}{l\,c} = c^{\left(\frac{1}{lc}\right)}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

or $M = (\text{base})^M$. (3), solved by successive approximations I find to give, to six decimals

$$c = 1.444667\ldots. = x \text{ max.},$$

for which value

$$y = \frac{1}{l\,c} = 2.\,71828\ldots. = e = \text{Nap. base.}$$

Now by reference to the first figure and construction of this article, we easily perceive that, if $C\ V$ cut the curve, then for any such point of intersection we must have $u = v$ or $v = \log v$ in the svstem whose basis is c.

Moreover, when $c = x$ max., $C\ V$ is tangent to the curve, giving

$$u = v = y = e = c^e = 1.444667^e\ldots.$$

Also noticing that if $c > 1.444667\ldots.$ $C\ V$ cannot intersect the curve; that for

$$c \gtrless {}^{1}_{1.444667\ldots.}$$

$C\ V$ intersects it in two points; and that for

$$c \leqq 1$$

$C\ V$ intersects it but once, we are prepared to make the following deductions:

I. *In logarithmic systems with bases greater than* $e^{\frac{1}{e}} = 1.444667\ldots.$ *there can be no logarithm equal to its antilogarithm or natural number.*

II. *In the system whose base is* $e\ \ = 1.444667\ldots.,$

$$\textit{if } e = \frac{1}{l\,c} \textit{ then } l\,c = \frac{1}{e};\ \textit{i. e. } c = e^{\frac{1}{e}},$$

there is but one such logarithm and its value is $2.718\ldots\ = e =$ *Nap. base, and it is at the same time the modulus* (M) *of the system,*

$$\textit{for } e = \frac{1}{l\ (1.444667\ldots.)}.$$

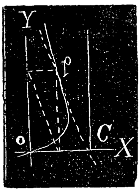
III. *In systems whose bases are less than unity, there is but one such logarithm and it is less than unity.*

Point of Inflection.—If we put

$$\frac{d^2 x}{d y^2} = 0 \text{ we have } y = \frac{2}{l c},$$

which substituted in

$$\frac{dx}{dy} \text{ gives } \frac{dx}{dy} = \tan \omega = - c^{1-\frac{2}{lc}};$$



also in the value of $x = \frac{y}{c^{y-e}}$ and we have $-\frac{x}{y} = - c^{1-\frac{2}{lc}}$.

$$\therefore \tan \omega = \left(-\frac{x}{y}\right)$$

for point of inflection. That is, *the tangent to the curve at the point of inflection is parallel to a diagonal of the rectangle formed by the co-ordinates of the point of inflection and the co-ordinate axes.*

SOLUTIONS OF PROBLEMS IN NO. 3.

Solutions have been received as follows: S.J. Child solved 12; Theo. L. DeLand solved 11; Prof. A. B. Evans solved 11, 12, 13 and I4; Prof. J. M. Greenwood solved 13 and 14; Henry Gunder solved 11 and 12; William Hoover solved 12; Artemas Martin solved 11, 12, 13 and 14; L, Regan solved 12; Walter Siverly solved 11, 12, 13 and 14; S. W. Salmon solved 11 and 12; Prof. J. Scheffer solved 11 and 12; and E. B. Seitz solved 15.

11. "Borrowed a sum of money at 8 per cent. simple interest and loaned it out again at 5 per cent. compound interest; in what time will I gain the amount borrowed."

SOLUTION BY HENRY GUNDER, GREENVILLE, OHIO.

Let the sum borrowed be \$1, and put t for the time to *gain* this sum. Then by the conditions we have $(1.05)^t = 2 + .08\,t$. By trial we find $t = 30$ years $+ x$, a fraction of a year. To find this fraction we have the following equation: $(1.05)^{30} + (1.05)^{30} \times .05\,x = 4.40 + .08\,x$, from which we find $x = .5735$ years; hence the time is 30y. 6m. 26d.

12. "Given the base $A\,C$ of a triangle and the ratio of $A\,B$ to $B\,C$ to find the locus of the point B by Geometry."